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Franco–American Meeting on the Mathematics of Random and Almost Periodic Potentials

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We present the abstracts of papers given at a conference held at the Ecole Polytechnique in June 1983 on the mathematics of random and almost periodic potentials.

From June 6–10, 1983, a conference was held at Ecole Polytechnique, sponsored by the USNSF under Grant No. INT82-12564, and by the CNRS as a French–American Joint Seminar. The conference was organized by the "authors" of this paper. The topic of the conference was the mathematical aspects of random and almost periodic potentials, that is, the study of a class of quantum Hamiltonians of the form $H = -\Delta + V$ on $L^2(\Omega^\nu, d^\nu x)$ and their discrete analogs, e.g., in one dimension

$$(hu)(n) = u(n+1) + u(n-1) + V(n)u(n) = \left[(h_0 + V)u\right](n) \qquad (1)$$

In both cases, V is not a fixed function; rather, V is an ergodic stochastic process. Explicitly, in the case of Eq. (1): Ω is a measure space with probability measure, μ, $T:\Omega\to\Omega$ is an ergodic transformation and f a function from $\Omega\to R$. Let $V_\omega(n) = f(T^n\omega)$ and $h_\omega = h_0 + V_\omega$. One is interested in properties of h_0 that hold for almost all ω, especially features of the spectrum, spectral measures, and eigenfunctions.

Two important examples are (1) the *random case* where $\Omega = \times_{n=-\infty}^{\infty}[a,b]$, $d\mu = \otimes_{n=-\infty}^{\infty} d\nu(x_n)$ with ν a measure on $[a,b]$ and $T(x)_n = x_{n+1}$ and $f(x) = x_0$ and (2) the *almost periodic case* where, as a typical example, Ω is a k-dimensional torus $\{(\theta_1,\ldots,\theta_k)\,|\,\theta \bmod 1\}$, $d\mu$ is the usual measure $(\times_{i=1}^{k} d\theta_i)$, $(T\theta)_i = \theta_i + \alpha_i$ where the α_i are rational, inde-

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pendent of each other and 1, and f is a continuous function on Ω. Some aspects of the theory are common to all ergodic processes, but most interesting features depend on detailed aspects such as whether the process is almost periodic or random or . . . .

From the physical point of view, case (1) appears in the physics of disordered solids, whereas case (2) appears in the physics of incommensurate systems, of bidimensional electrons in a periodic potential and an orthogonal magnetic field, and of lattices of supraconducting wires.

Below, we first list the program with complete addresses for readers wishing more information, and then we give a set of "abstracts" provided by the speakers (with a common bibliography appearing at the end of the paper). Some additional references are provided alphabetically following the references for the abstracts.

LIST OF SPEAKERS AND ADDRESSES

Monday, June 6

W. Craig *Department of Mathematics 253-37, California Institute of Technology, Pasadena, California 91125:* Pure point spectrum for discrete almost periodic Schrödinger operators.

J. Bellissard *Centre de Physique Theorique, CNRS, Luminy-Case 907, F-13288 Marseille Cedex 9, France:* Small divisors effects in quantum mechanics.

T. Spencer *Courant Institute, 251 Mercer St., New York, New York 10012:* Absence of diffusion in the Anderson tight binding model.

Tuesday, June 7

R. Lima *Centre de Physique Theorique, CNRS, Luminy-Case 907, F-13288 Marseille Cedex 9, France:* Spectral properties of the almost Mathieu Hamiltonian.

P. Sarnak *Courant Institute, 251 Mercer St., New York, New York 10012:* Complex almost periodic potentials.

R. Prange *Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742:* A solvable almost periodic Schröedinger equation with a singular continuous spectrum.

B. Simon *Department of Mathematics 253-37, California Institute of Technology, Pasadena, California 91125:* m Functions and the absolutely continuous spectrum of one-dimensional almost periodic operators.

D. Bessis *Service de Physique Theorique, CEN Saclay, 91191 Gif-sur-Yvette Cedex, France:* Green function behavior for a Hamiltonian with purely singular spectrum.

M. Kohmoto *Department of Physics, University of Illinois, Urbana, Illinois 61801:* Dynamical map related to an almost periodic model.

Wednesday, June 8

Y. Guivarc'h *Department of Mathematics, University of Rennes, Rennes, France:* Products of random matrices.

H. Kesten *Department of Mathematics, Cornell University, Ithaca, New York 14853:* Convergence in distribution of products of random matrices.

J. Poschel *Mathematik, ETH-Zentrum, CH-8092 Zurich, Switzerland:* An extension of a result by Dinaburg and Sinai.

M. Mehta *Service de Physique Theorique, CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France:* Random matrices—A review.

R. Carmona *Department of Mathematics, University of California, Irvine, California 92717:* One-dimensional random Schrödinger operators—A survey.

J. Lacroix *UER Mathematiques et Informatique, Campus de Beaulieu, 35042 Rennes Cedex, France:* Pure point spectrum for the limit difference Schrödinger equation in a strip.

Thursday, June 9

F. Wegner *Institut für Theoretische Physik, Ruprecht-Karls-Universität, D-6900 Heidelberg, Federal Republic of Germany:* Anderson transition and nonlinear σ model.

B. Souillard *Centre de Physique Theorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France:* The study of a mobility edge—The Anderson model on the Bethe lattice.

F. Delyon *Centre de Physique Theorique, Ecole Polytechnique, 91128 Palaiseau Cedex, France:* The density of states and the rotation number for finite difference equations and their properties.

J. Avron *Department of Physics, The Technion, 32 000 Haifa, Israel:* Topological invariants for periodic matrices.

D. Thouless *Department of Physics, University of Washington, Seattle, Washington 98195:* Quantized Hall conductance, matter transport, and band gaps in quasiperiodic systems.

G. Toulouse *Groupe de Physique des Solides, Ecole Normale Superieure, 24 rue Lhomond, F-75005 Paris, France:* Edge states in quantized Hall conductance and random walks on percolation clusters.

Friday, June 10

I. Herbst *Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903:* One-dimensional systems in an electric field.

F. Bentosela *Centre de Physique Theorique, CNRS, Luminy-Case 907, F-13288 Marseille Cedex 2, France:* Electrons in a solid submitted to an electric field.

R. Johnson *Department of Mathematics, Universität Heidelberg, D-6900 Heidelberg 1, Federal Republic of Germany:* Almost periodic spectral problems and nonlinear evolution equations.

ABSTRACTS OF THE TALKS

J. Avron, R. Seiler and B. Simon, *Topological invariants for periodic matrices:* We described the geometric and homotopic significance of the quantised conductances in the (normal) Hall effect. See Refs. 1, 81, and 83.

J. Bellissard, *Small divisors effects in quantum mechanics:* In several examples of quantum mechanical systems, the perturbation theory exhibits small divisors. This occurs for almost periodic potentials and time-dependent periodic Hamiltonians. We show how to avoid the divergencies by using the Kolmogoroff–Arnold–Moser[7,19,61,77] procedure adapted to the problem. It actually works only for a few number of examples and it would need an extension to treat the problem in full generality.[2,3,5,6,14,67]

In the region of energy corresponding to strong resonances the previous procedure does not work. In quantum mechanics this is due to the possibility of long-range tunneling effects in the Fourier space. We argue that such effects occur. In this case the wave function should be extended and even chaotic.[4,28,44,65] It should actually be localized on an unbounded sequence of points in Fourier space, strongly sensitive to the value of the energy. The corresponding spectral measure should be singular continuous.[2]

We also show an example of almost periodic Hamiltonian,[4] which has a Cantor spectrum of zero Lebesgue measure, and eigenstates which are likely to be chaotic in the large, with a sensitive dependence with respect to the energy.

F. Bentosela, *Electrons in a solid submitted to an electric field:* (1) We prove that the Schrödinger operator in one dimension corresponding to a particle in a solid (amorphous or crystalline) submitted to a constant external electric field has purely absolutely continuous spectrum. (2) In the case of the semi-infinite crystal we calculate numerically the electrical field dependence of the resonances and show that they oscillate strongly. The same study has been undertaken in the random case; its goal is now to study the dependence of the resonance width with respect to the disorder. See Refs. 8–10.

D. Bessis, J. Geronimo, and P. Moussa, *Green function behavior for a Hamiltonian with purely singular spectrum*[11]: We analyze the end point spectrum of the quadratic map Hamiltonian[4] which is thought to be a paradigmatic example of a Schrödinger operator with purely singular continuous spectrum. This is done by introducing the Mellin transform of the density of states, which is shown to be a meromorphic function in $\mathbb{C}$ with poles on a semi-infinite rectangular lattice. While simple scaling arguments propose for the density of states a formula of the type:

$$\sigma(E) \sim \bar{C}(E_{\text{end}} - E)^{\delta}$$

Where δ is the spectral dimension, this formula must be replaced in the presence of a singular spectrum by

$$\sigma(E) \sim (E_{\text{end}} - E)^{\delta}\left\{\bar{C} + \sum_{k=1}^{\infty} C_{\kappa}\cos\left[\kappa\tau\ln(E_{\text{end}} - E) + \phi_{\kappa}\right]\right\}$$

which shows that the "constant" $\bar{C}$ has no limit when $E \to E_{\text{end}}$, but presents infinite bounded oscillations. The same results hold for mechanical systems on fractal structure such as the Sierpinsky Gasket model, for instance.[71]

For the algebraic hierarchical models of statistical mechanics,[17] for which the renormalization group reduces to a rational transformation, our analysis leads to the introduction of an imaginary part to be added to the usual critical index to describe the critical behavior: this has for consequence the existence of an oscillatory behavior of the critical amplitudes.

R. Carmona, *One-dimensional random Schrödinger operators—A Survey:* The talk begins with a quick review of the time-honored O.D.E. (ordinary differential equation) approach to the study of the spectral properties of one-dimensional deterministic Schrödinger operators (Titchmarsch–Weyl *m* functions, eigenfunction expansions for Sturm–Liouville problems, Plancherel formula, . . .)

Then the works done during the last decade in the random case are reviewed. This amounts to studying the almost sure spectral characteristics of Schrödinger operators with stationary ergodic random potentials. We concentrate on (i) *thermodynamic properties* such that the existence and the regularity of the density of states, the attraction-repulsion problem for energy levels and central limit theorems and large deviation results for the density of states; (ii) *spectral types*: presence or absence of an absolute component in the spectrum, exponential localization when the spectrum is dense pure point, perturbations by deterministic potentials (for example, a constant electric field); (iii) the numerous *problems* still *open*.

W. Craig, *Pure point spectrum for discrete almost periodic Schrödinger operators:* The discrete Schrödinger operator on Z^ν is considered in this talk:

$$(H\psi)_j = \sum_{|k|=1} \psi_{j+k} + \frac{1}{\epsilon} q_j \psi_j = \lambda \psi_j \tag{1}$$

We consider sequences q_j that are almost periodic in the sense that there exists a "reasonable" function $Q(x)$ on the torus T^ν such that for some rationally independent ν vectors ω:

$$q_j = Q(\omega j) \tag{2}$$

"Reasonable" is a technical condition, which for $\mathbb{Z}^1$ is taken to be "of bounded variation" (this implies that at least q_j is l^2–almost periodic).

Examples of such potentials with entirely pure point spectrum are constructed for ϵ sufficiently small. The method is an inverse spectral procedure, with a rapidly convergent iteration scheme to overcome small divisors similar to the method of Kolmogorov, Arnold, and Moser. Input is a spectral generating sequence d_j satisfying (2) and as well a nonresonance condition.

$$|d_j - d_k| = |D(\omega j) - D(\omega k)| \geqslant c|j-k|^{-\tau} \tag{3}$$

The potential q_j is constructed along with a unitary operator G such that

$$G^{-1}HG = D \equiv d_j \delta_{ij}$$

and

$$|g_{ij} - \delta_{ij}| < \epsilon \exp(-\rho|i-j|)$$

Examples are given in which the spectrum is an interval, and in which the spectrum is a Cantor set of any fractal dimension from 0 to 1. See Refs. 5, 14, 38, and 67.

F. Delyon, *The density of states and the rotation number for finite difference equations and their properties:* For a continuous Schrödinger equation with an almost periodic potential, it was proved[38] that the integrated density of states $k(\lambda)$, which is equal to twice the rotation number $\alpha(\lambda)$, lies in the frequency module of the potential when λ is outside the spectrum; this yielded a labeling of the gaps of the spectrum. Except in the special cases where C^* algebra technics give the result, the question was opened whether an analogue does hold for Jacobi matrices acting on $l^2(\mathbb{Z})$,

$$(Hu)(n) = -u(n+1) - u(n-1) + V(n)u(n)$$

and more generally for second-order finite difference equations.

We discussed[16] a rotation number $\alpha(\lambda)$ for finite difference equations. The integrated density of states $k(\lambda)$ is related to it by $k(\lambda) = 2\alpha(\lambda)$. For almost periodic operators, $k(\lambda)$ is proved to lie in the frequency module whenever λ is outside the spectrum. The homotopic invariants appearing there are related to the value of the quantized Hall effect of a two-dimensional electron in a periodic potential and a strong orthogonal magnetic field.

J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model:* We prove the absence of diffusion[23] and the existence of a dense family of eigenstates (with probability 1) for the Hamiltonian

$$H = -\Delta + \lambda v \qquad \text{on } \mathbb{Z}^d, \quad d \geqslant 1$$

provided that either λ is a large or that the energy lies in the band tails. The potential v_j are assumed to be independent, mean zero random variables with a bounded distribution. Our proof is based on perturbation theory about an infinite sequence of block Hamiltonians and is related to KAM methods. See also Ref. 24.

Y. Guivarc'h, *Products of random matrices:* We described the known results about random matrix products in the following directions: characteristic exponents, limit theorems (independent case), stability properties of the characteristic exponents. In the following references others can be found: 18, 25, 26, 29, 30, 31, 32, 33, 34, 36, 40, 49, 52, 53, 54, 55, 64, 70, 75, 76, 86, 87, and 88.

I. Herbst, *One-dimensional systems in an electric field:* In this talk I presented a simplified version of joint work with J. S. Howland[35] dealing with a one-dimensional system (periodic, random, etc.) in an electric field. The work shows that under certain analyticity assumptions on the potential, it is possible to discuss resonances as poles of resolvent matrix elements.

In addition, I speculated about the time development of a localized random system after the electric field is turned on.

R. Johnson, *Almost periodic spectral problems and nonlinear evolution equations:* We discussed the discrete rotation number, its reduction to the continuous rotation number via the suspension procedure, and the connection with C^*-algebra theory. We also posed two problems, one having to do with nonlinear evolution equations and the other with numerical evaluation of Lyapounov exponents.

H. Kesten and F. Spitzer, *Convergence in distribution of products of random matrices:* We discussed conditions for convergence of the distribution of $M_n := A_1 A_2 \ldots A_n$, where the A_i, $i \geqslant 1$ are independent, identically distributed *nonnegative* matrices. The limit distribution should not be concentrated on the zero matrix. Note that the M_n are not normalized. It turns out that such convergence is also equivalent to the equality

$$\log(\text{largest eigenvalue of} \langle A_1 \rangle) = \lim_{n\to\infty} \frac{1}{n} \log \|M_n\| \qquad (1)$$

The almost sure existence of the limit in the right-hand side was proved in Refs. 26 and 41. For nonegative A_i one always has the left-hand side of (1) $\geqslant$ the right-hand side of (1), and usually this is a strict inequality.

M. Kohmoto, *Dynamical map related to an almost periodic model:* A class of the (discrete) almost periodic Schrödinger equation is related to a three-dimensional volume-preserving map.[42,44] An existence of a conserved quantity actually restricts it on a two-dimensional manifold. The potentials are constant except for steps at special points.

There are various types of cycles.[39] Each represents some scaling behavior of the system. In particular, a fixed-point analysis of the six cycle develops the scaling for the spectrum found numerically.[43] The heteroclinic points of the map explain the infinite hierarchical structure of the energy spectrum which is likely to be a Cantor set.[45]

J. Lacroix, *Pure point spectrum for the limit difference Schrödinger equation in a strip:* Let H be the finite difference Schrödinger operator with a random potential in a strip of width d. We suppose that the sequence of potentials is a family of independent random variables with a common law, absolutely continuous of $\mathbb{R}$. In the case $d = 1$, it was known that the spectrum is almost surely pure point, with exponentially decaying eigenvectors (see Ref. 47 or 49). We establish here the same result for all d, using the theory of operators associated with the action of the symplectic group on some compact boundary appearing in classical symplectic geometry. [It was already known that the strict positivity of the smallest positive Lyapounoff exponent, associated with a random product of symplectic matrix, implies absence of absolutely continuous spectrum for H (see Ref. 50).] (See also Ref. 51.)

R. Lima, *Spectral properties of the almost Mathieu Hamiltonian:* This seminar is a report on a joint work with J. Bellissard and D. Testard.[6] We study some properties of the spectrum of operators of the form

$$H_x^{(\mu)}\psi(n) = \psi(n+1) + \psi(n-1) + \mu V(x - n\theta)\psi(n)$$

where $\psi \in l^2(\mathbb{Z})$, V is a continuous function on the circle Π, θ is an irrational number, $x \in \Pi$, and μ is a real positive number (the coupling constant). The case where $V(y) = 2\cos \pi y$ (almost Mathieu potential) is particularly interesting from our point of view because in that case it is possible to relate the spectral properties for small μ with spectral properties for large μ, using a duality argument.

First we use the Kolmogorov–Arnold–Moser recursion process in order to treat the small divisor problem which naturally arises in this problem. Then the existence of absolutely continuous spectrum for small coupling is proved, together with an estimation of the Lebesgue measure of this part of the spectrum, showing that the measure is positive. The result is true for all $x \in T$ and a subset of θ of full measure. Specifying θ to be a Roth number with special type of continued fraction ($\limsup_{n>\infty} a_n = +\infty$), then for any $\epsilon > 0$ there is a $\lambda_0 > 0$ such that, for almost all $x \in T$ and $\mu \leqslant \lambda_0$ the absolutely continuous part of the spectrum of the almost Mathieu Hamiltonian has a Lebesgue measure greater than $4 - \epsilon$ (remark that $4 + 4|\mu|$ is a trivial upper bound of the measure of the spectrum). Still such θ have full measure. Passing from small coupling constant to large coupling constant we prove the existence of an infinite number of eigenvalues whose closure has positive Lebesgue measure. The corresponding eigenvectors have exponential decay. For the special class of θ described above we can give an improvement on the size of the closure of the set of eigenvalues, since in that case we prove that the Lebesgue measure is greater than $(4 - \epsilon)\mu$.

M. Mehta, *Random matrices—A review:* The review articles or books on the topic are as follows. Some of them have a considerable list of references (e.g., Ref. 13 refers to more than 300 articles). See Refs. 12, 13, 20, 27, 56, 57, 58, 59, 60, 66, and 91].

J. Moser and J. Pöschel, *An extension of a result by Dinaburg and Sinai:* We consider the stationary Schrödinger equation $(*)$ $Ly = -y'' + q(x)y = \lambda y$ on the real line, where q is quasiperiodic with basic frequencies $\omega = (\omega_1, \ldots, \omega_d)$. The result of Dinaburg and Sinai[19,77] is extended in the following way.

If $\mu = (k, \omega)/2$ is sufficiently large and badly approximable by all other resonances $(j, \omega)/2$, $j \neq k$, then the spectral gap $\alpha^{-1}(\mu)$ (α the rotation number) is generally open, and $(*)$ has solutions $e^{i\mu x}(\chi_1 + x\chi_2)$, $e^{-i\mu x}\chi_3$ at the endpoints of the gap, where χ_1, χ_2, χ_3 are quasiperiodic with basic frequencies ω. If the gap is collapsed, then $\chi_2 = 0$.

These gaps cluster at the points in the absolutely continuous spectrum provided by the theorem of Dinaburg and Sinai. Also see Ref. 38.

R. Prange, *A solvable almost periodic Schröedinger equation with a singluar continuous spectrum:* The papers of J. Bellissard, P. Sarnak, and M. Kohmoto at this symposium are particularly closely related to this work. For references see 22, 28, 68, and 69.

P. Sarnak, *Complex almost periodic potentials:* Spectral properties of Schrödinger operators of the type $H = -\Delta + \epsilon V$, where Δ is the Laplacian, V a quasiperiodic potential, and ϵ a coupling constant, are developed. V is taken to be finite sum of exponentials with generic frequencies. For small ϵ a strong stability is shown. On the other hand, examples (in the finite difference case) are given, for which a transition in the type of spectrum occurs, as is increased. The talk is based on the author's paper.[78]

B. Simon, *m Functions and the absolutely continuous spectrum of one-dimensional almost periodic operators:* Discussion of the results in the following five papers: Refs. 15, 38, 46, 62, and 80.

B. Souillard, *The study of a mobility edge: The Anderson model on the Bethe lattice:* We consider the Anderson model on the Bethe lattice. It is proved[48] for a large class of distributions of potential that with probability 1:

(i) For large disorder, the spectrum is pure point with exponentially decaying wave functions and the static conductivity vanishes.

(ii) For small disorder, the spectrum is purely absolutely continuous for small energies whereas it is pure point with exponentially decaying wave functions and the static conductivity vanishes for energies large enough.

(iii) The density of states is an analytic function of the energy at the mobility edges.

(iv) The localization length, if measured with the natural distance on the Bethe lattice, diverges with the critical exponent $\nu = 1$. This result implies $\nu = 1/2$ for the Anderson model in large enough dimension.

These results provide the first model where the Anderson transition is proved and the first exact critical exponent for the localization problem.

D. Thouless, *Quantized Hall conductance, matter transport and band gaps in quasiperiodic systems:* An account was given of the theory of the Hall effect in a two-dimensional periodic potential, in which sub-bands can be characterized by a topological invariant proportional to the Hall conductance.(83) The same invariant describes the integrated current carried by a potential which is the sum of two periodic potentials, one of which is moved slowly relative to the other.(81) In the case of incommensurate potentials, or a two-dimensional potential incommensurate with the flux lattice, this gives an extra physical significance to the integers used to characterize the energy gaps.(38)

G. Toulouse, *Edge states in quantized Hall conductance and random walks on percolation clusters:* Solving the Schrödinger equation on various structures, regular lattices, fractal structures, . . . with or without an applied magnetic field, one is able to treat a number of applications: superconducting diamagnetism, Landau levels, diffusion. Two such applications were chosen for discussion. For references, see 72, 73, 74, 84, and 85.

F. Wegner, *Anderson transition and nonlinear σ model:* The behavior of a quantum mechanic particle moving in a random potential is considered. Symmetry arguments are given which allow the mapping of such a system onto a field theoretic model of interacting matrices. This model yields an expansion of the critical exponents at the mobility edge around the lower critical dimensionality two. For references, see 21, 37, 63, 79, 82, 89, and 90.

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